LAMINAR FLOW IN A UNIFORMLY POROUS CHANNEL

A THESIS

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PREFACE

The problem of laminar flow in channels with porous walls was first introduced to Mr. F. M. White, Jr.† and the author in the graduate fluids course conducted in the Mechanical Engineering Department of the Georgia Institute of Technology by Dr. M. J. Goglia. They were thoroughly briefed on the literature then existing that entertained the problem and were given the benefit of Dr. Goglia's experience with it.

Under the guidance of Dr. Goglia, they began to pursue the problem. There were present many formidable difficulties, as had been pointed out in previous literature, until Mr. White was struck with the particular manner in which the free parameter "wall Reynolds number" appeared in the differential equation associated with the solution of the problem. After a careful analysis and much discussion, Mr. White recognized a transformation that reduced the differential equation from the intractable to the difficult.

As is usual in joint scientific efforts, the initial progress upon this problem was mainly accomplished through the medium of numerous stimulating and enthusiastic discussions between the two investigators. However, once the transformation of the differential equation was effected, the path to completion of the problem became clearly defined. The path pursued in the work reported in this thesis was the investigation of the

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†This thesis is presented in conjunction with the Ph.D. thesis of Mr. Frank M. White, Jr., presented to the Mechanical Engineering Department of the Georgia Institute of Technology.

series solution associated with the differential equation and the subsequent reduction of this strange series to summation form, whereas Mr. White carried out the computer solutions and general analysis.

The work reported in this thesis was under the auspices of Mr. White.

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SUMMARY

This investigation was conducted primarily to obtain a series solution of the differential equation determining the velocity profile, pressure distribution, and friction coefficient, as a function of the wall Reynolds number, for fluids flowing in channels and pipes with porous walls.

Various discussions of this problem have appeared in the literature from time to time, as will be pointed out in the context of this thesis. The method of attack employed by previous authors was the perturbation solution. This approach is very limited, since a solution is obtained over an extremely small range of wall Reynolds number.

In this work the problem is presented in its entirety for the case of uniformly porous channels. In addition, the series solution is given for the semi-porous channel (i.e. one wall of the channel porous) and for circular pipes.

The coefficients have been investigated by Mr. White on the I.B.M. 650 computer, and this thesis contains a comparison of the computer values with the series values. The results are gratifying.

The velocity distribution, pressure distribution, and friction coefficient follow readily when use is made of the definition of a stream function. These are presented in Mr. White's thesis. An anomaly appears in the pressure distribution investigation, since in some cases the pressure actually rises as the fluid flows down the pipe. This, of course, is a consequence of the slowing down of the fluid.

The series arising from the differential equation is a strange one indeed. This is apparent when it is noticed that each of the constants appearing in the solution (which is an infinite series) is a series itself. The proof of the convergence of the series is itself very unusual.

CHAPTER I

INTRODUCTION

For purposes of mathematical attack, a porous pipe is thought of as being perfectly porous, i.e., the flow is uniformly and evenly distributed through the walls.

The practicality of the porous wall pipe is possibly best thought of in relation to gaseous diffusion processes, though a multitude of other applications can be envisioned, such as transpiration cooling and boundary layer control.

The problem was apparently first introduced by Olson (1)† in 1949, but the greatest contribution to date was made by Berman (2), by virtue of the introduction of a stream function and the reduction of the physical problem to a differential equation. Extensions to the work of Berman have been made by Yuan (3), Sellars (4) and Donoughe (5). Yuan arrived at the differential equation for the case of pipe flow, Donoughe obtained the differential equation for semi-porous channels, and Sellars applied a perturbation analysis to high suction Reynolds number flows.

The purpose of this investigation was to solve the differential equations arising in each of these cases by virtue of series methods and to compare the IBM computer solutions to the series solutions.

The series arising from each of the geometries is found to be quite different (as might be expected) and therefore a separate analysis

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†Numbers in parentheses following authors' names refer to the Bibliography.

is necessary for each case. Only the channel (fully porous) is detailed up to the point where the differential equation is obtained, but the differential equation for all cases is presented and the series solution is carried out in detail for each. A detailed analysis of other geometries is included in the unpublished thesis of Mr. White (6).

CHAPTER II

THE POROUS WALL CHANNEL

Assumptions.--In order to resolve the problem, the following assumptions were made:

(1) The walls are uniformly porous.

(2) The flow is steady with time.

(3) The fluid is incompressible.

(4) The flow is viscous and laminar.

(5) The flow is two-dimensional (i.e. there are no variations with z).

(6) No body forces are present.

Geometry.--A rough sketch of the geometry is given below.

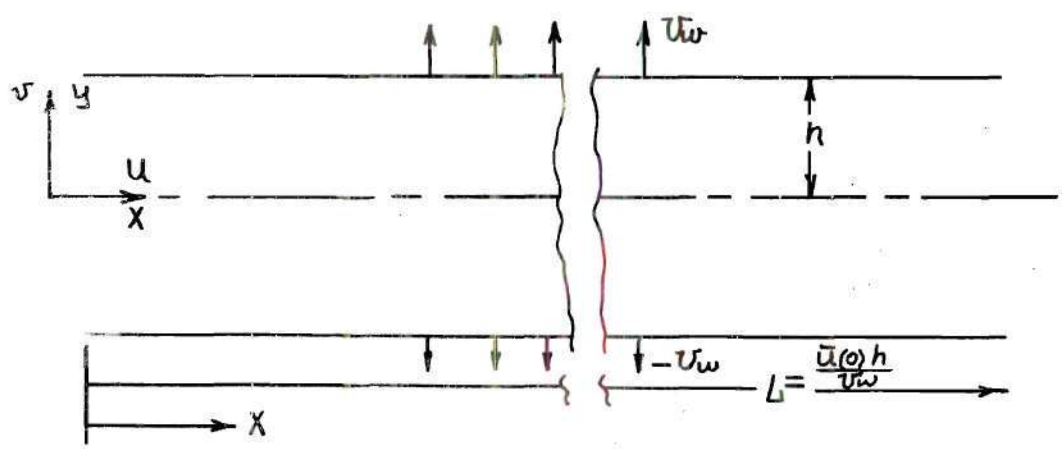

Figure 1.

The Navier-Stokes Equations.--The Navier-Stokes Equations in rectangular Cartesian coordinates for incompressible flow are given by:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = X - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = Y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = Z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

and the equation of continuity is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

Reduction of the equations.--From the assumptions of steady flow:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} \equiv 0$$

and the assumption of two-dimensionality yields:

$$\frac{\partial p}{\partial z} = \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial z^2} = \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} \equiv 0$$

This, of course, eliminates equation (3).

The assumption of no body forces yields:

$$X = Y = Z \equiv 0$$

Non-dimensionalizing the differential equations.--At this point it appears convenient to non-dimensionalize the y-coordinate in terms of the channel width.

Define:

$$\lambda \equiv y/h$$

then,

$$dy = h d\lambda$$

This definition allows the Navier-Stokes equations to be transformed from the x-y plane to the x-λ plane. With the above simplifications and the transformation identity, equations (2), (3), and (4) then become:

$$u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial \lambda} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 u}{\partial \lambda^2}\right) \tag{5}$$

$$u\frac{\partial v}{\partial x} + \frac{v}{h}\frac{\partial v}{\partial \lambda} = -\frac{1}{\rho h}\frac{\partial p}{\partial \lambda} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial \lambda^2}\right) \tag{6}$$

$$\frac{\partial u}{\partial x} + \frac{1}{h}\frac{\partial v}{\partial \lambda} = 0 \tag{7}$$

The boundary conditions are given by:

(A) $u(x,\pm 1) = 0$--from the no-slip condition.

(B) $v(x,0) = 0$--from symmetry.

(C) $v(x,\pm 1) = \pm V_w$ = constant--uniformly porous walls.

(D) $\left.\frac{\partial u}{\partial \lambda}\right)_{\lambda=0} = 0$--from symmetry.

One of the first steps toward the solution of the problem was the introduction of a stream function $\psi(x,\lambda)$ by Berman (2) in order to satisfy the equation of continuity, and the subsequent postulation of a Bernoulli product solution of the form:

$$\psi(x,\lambda) = g(x) \cdot \phi(\lambda) \tag{8}$$

From the definition of a stream function, the velocity in the x-direction is given by:

$$u = \frac{\partial\psi}{\partial y} = \frac{g(x)}{h} \cdot \phi'(\lambda) \tag{9}$$

and in the y-direction by:

$$v = -\frac{\partial\psi}{\partial x} = -g'(x)\phi(\lambda) \tag{10}$$

At this point another important and interesting analysis must be carried out. The boundary conditions will be examined in the light of the new postulation. Employing the boundary conditions and equations (9) and (10), it is found that the following conditions prevail:

From boundary condition (B),

$$v(x,0) = 0 = - g'(x) \cdot \phi(0)$$

or since $g'(x) = 0$ would lead to the trivial conclusion that the y-component of velocity was everywhere zero, it must be concluded that:

$$\phi(0) = 0 \tag{11}$$

Using boundary condition (C),

$$v(x,\pm 1) = \pm v_w = \text{constant} = - g'(x)\phi(\pm 1)$$

This is a most interesting condition after noting that $\phi(\pm 1)$ is a real number and also that $\phi(\pm 1)$ is not equal to zero, since it would yield an incompatible solution if it were. With this foremost in mind, the meaning of this boundary condition will be investigated.

The result of boundary condition (C) can be written in the form:

$$g'(x) = \frac{-v_w}{\phi(1)} = \text{constant} = C_1 \tag{12}$$

Upon integrating this yields:

$$g(x) = C_1 x + C_2 \tag{13}$$

Since $g(x) = g(0)$ at $x = 0$, equation (13) can be written as:

$$g(x) = C_1 x + g(0) \tag{14}$$

also noting that C_1 is given in equation (15), permits equation (14) to be written as:

$$g(x) = g(0) - \frac{v_w}{\phi(1)} x \tag{15}$$

The term $g(0)$ can be evaluated from a consideration of the stream function at the entrance as follows:

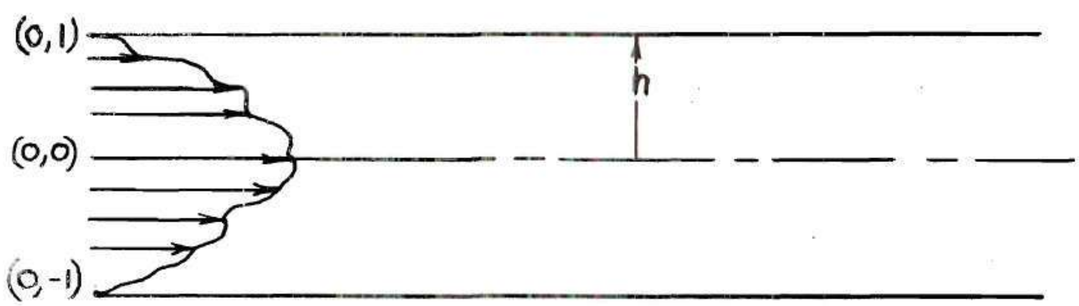

Figure 2.

The definition of a stream function yields the information that:

$\psi_2 - \psi_1$ = volume rate of flow between points 1 and 2.

Similarly,

$\psi(0,1) - \psi(0,0)$ = volume rate of flow between (0,1) and (0,0).

Now since the volume rate of flow (per unit depth) is given by:

$$h \cdot \bar{u}(0) \cdot 1$$

where $\bar{u}(0)$ is the average velocity at the entrance, then,

$$\psi(0,1) - \psi(0,0) = h \cdot \bar{u}(0) \tag{16}$$

Since it has been assumed that:

$$\psi(x,\lambda) = g(x) \cdot \phi(\lambda)$$

it follows that:

$$\psi(0,1) - \psi(0,0) = g(0)\phi(1) - g(0)\phi(0)$$

It has also been shown that $\phi(0) = 0$, therefore,

$$g(0) = \frac{\psi(0,1) - \psi(0,0)}{\phi(1)}$$

or using equation (16),

$$g(0) = \frac{h\,\bar{u}(0)}{\phi(1)} \tag{17}$$

Returning to equation (15) and substituting equation (17) yields:

$$g(x) = \frac{1}{\phi(1)}\left[h\,\bar{u}(0) - v_w x\right] \tag{18}$$

The velocity components are now given by:

$$u = \frac{g(x)}{h}\cdot\phi'(\lambda) = \left[\bar{u}(0) - \frac{v_w x}{h}\right]\frac{\phi'(\lambda)}{\phi(1)} \tag{19}$$

$$v = -\,g'(x)\,\phi(\lambda) = -\left[\frac{-v_w}{\phi(1)}\right]\cdot\phi(\lambda) = v_w\,\frac{\phi(\lambda)}{\phi(1)} \tag{20}$$

Reduction of the differential equation.--For convenience, define:

$$f(\lambda) = \frac{\phi(\lambda)}{\phi(1)}$$

then the velocities can be written as:

$$u = \left[\bar{u}(0) - \frac{v_w x}{h}\right]f'(\lambda)$$

$$v = v_w f(\lambda)$$

Substitution of the above values into the Navier-Stokes equations, i.e. equations (5) and (6), yields:

$$\left\{\left[\bar{u}(0) - \frac{v_w x}{h}\right] f'(\lambda)\left(\frac{-v_w}{h}\right) f'(\lambda)\right\} + \left\{\left(\frac{v_w}{h}\right) f(\lambda)\left[\bar{u}(0) - \frac{v_w x}{h}\right] f''(\lambda)\right\} =$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left\{0 + \left[\bar{u}(0) - \frac{v_w x}{h}\right]\frac{f'''(\lambda)}{h^2}\right\} \tag{21}$$

$$\left\{\left[\bar{u}(0) - \frac{v_w x}{h}\right] f'(\lambda) \cdot [0]\right\} + \left\{\left(\frac{v_w}{h}\right) f(\lambda)(v_w) f'(\lambda)\right\} = \tag{22}$$

$$= -\frac{1}{\rho h}\frac{\partial P}{\partial \lambda} + \nu\left\{0 + \frac{1}{h^2} v_w f''(\lambda)\right\}$$

Upon simplifying, the equations become:

$$\left(\frac{v_w}{h}\right)\left[\bar{u}(0) - \frac{v_w x}{h}\right]\left\{- f'(\lambda)^2 + f(\lambda) f''(\lambda)\right\} = \tag{23}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \left[\bar{u}(0) - \frac{v_w x}{h^2}\right]\frac{\nu}{h^2} f'''(\lambda)$$

$$\frac{v_w^2}{h} f(\lambda) f'(\lambda) = -\frac{1}{\rho h}\frac{\partial P}{\partial \lambda} + \frac{v_w}{h^2} f''(\lambda) \tag{24}$$

Berman (2) first noticed that equation (24) was a function of λ only, and that this would permit the following very important conclusion:

$$\frac{\partial^2 P}{\partial x \partial \lambda} \equiv 0 \equiv \frac{\partial^2 P}{\partial \lambda \partial x} \tag{25}$$

Differentiating equation (23) with respect to λ yields:

$$\frac{v_w}{h}\left[\bar{u}(0) - \frac{v_w x}{h}\right]\frac{\partial}{\partial \lambda}\left\{-f'(\lambda)^2 + f(\lambda) f''(\lambda)\right\} = -\frac{1}{\rho}\frac{\partial^2 P}{\partial x \partial \lambda} + \tag{26}$$

$$+ \left[\bar{u}(0) - \frac{v_w x}{h}\right]\frac{\nu}{h^2}\frac{\partial}{\partial \lambda}\left[f'''(\lambda)\right]$$

Making use of equation (25) and cancelling the term,

$$\left[\bar{u}(0) - \frac{v_w x}{h}\right]$$

yields:

$$\frac{v_w}{h}\frac{\partial}{\partial\lambda}\left\{-f'(\lambda)^2 + f(\lambda)f''(\lambda)\right\} = \frac{\nu}{h^2}\frac{\partial}{\partial\lambda}\left[f'''(\lambda)\right] \qquad (27)$$

or,

$$\frac{\partial}{\partial\lambda}\left\{\left(\frac{v_w h}{\nu}\right)\left[f'(\lambda)^2 - f(\lambda)f''(\lambda)\right] + f'''(\lambda)\right\} = 0 \qquad (28)$$

Defining a "wall Reynolds number",

$$R \equiv \frac{v_w h}{\nu}$$

and integrating yields:

$$R\left[f'(\lambda)^2 - f(\lambda)f''(\lambda)\right] + f'''(\lambda) = K = \text{Constant} \qquad (29)$$

Equation (29), which describes the flow in the channel, is a third order non-linear differential equation with parametric coefficients. It has four boundary conditions, the fourth condition determining the constant K. K must satisfy the parametric quality of the equation (i.e. it must vary in some manner as R varies). From this consideration it seems evident that the wall boundary condition, $v = v_w$, determines K, and K is actually a function of v_w or R.

The boundary conditions can be written as:

$$u(x,1) = \left[\bar{u}(0) - \frac{v_w x}{h}\right] f'(1) = 0$$

Therefore,

$$f'(1) = 0.$$

$$v(x,0) = v_w\, f(0) = 0$$

and

$$f(0) = 0.$$

$$\frac{\partial u}{\partial \lambda}(x,0) = \left[\bar{u}(0) - \frac{v_w x}{h}\right] f''(0) = 0$$

and

$$f''(0) = 0.$$

$$v(x,1) = v_w = v_w\, f(1)$$

and

$$f(1) = 1.$$

To sum up, the boundary conditions are:

$$f'(1) = 0 \tag{30}$$

$$f(0) = 0$$

$$f''(0) = 0$$

$$f(1) = 1$$

and the differential equation is:

$$R\left[f'(\lambda)^2 - f(\lambda)f''(\lambda)\right] + f'''(\lambda) = K = K(R) = \text{Constant} \qquad (31)$$

The differential equation and associated boundary conditions defy an exact solution when written in this form. The differential equation has been attacked by the method of perturbations in the literature. The perturbation solutions have an extremely small radius of convergence.

CHAPTER III

THE EXACT SOLUTION OF THE DIFFERENTIAL EQUATION

Transformation of the differential equation.--The solution of the differential equation describing the flow, i.e., equation (31), in its present form is probably impossible. However, once the observation is made that the parameter R can be transformed out of the equation and into the boundary conditions, the equation is reduced to a very difficult, but nevertheless solvable, fourth order, non-linear, differential equation.

The general solution of equation (31) for arbitrary values of R proceeds as follows:

Rewriting equation (31) for convenience,

$$R\left[f'(\lambda)^2 - f(\lambda)f''(\lambda)\right] + f'''(\lambda) = K(R) \tag{31}$$

The first step is to differentiate in order to eliminate the constant, which yields:

$$R\left[f'(\lambda)f''(\lambda) + f''(\lambda)f'(\lambda) - f(\lambda)f'''(\lambda) - f'(\lambda)f''(\lambda)\right] + f''''(\lambda) = 0$$

or

$$R\left[f'(\lambda)f''(\lambda) - f(\lambda)f'''(\lambda)\right] + f''''(\lambda) = 0 \tag{32}$$

The solution of equation (32), as it stands, is without question very difficult. Possibly the only method of solution is through the medium of perturbation extensions of the limiting solutions now known.

If, however, it is noticed that the wall Reynolds number is a similarity parameter, a transformation can be defined that will cause the free parameter R to vanish from the equation.

The transformation which effects this simplification is defined by:

$$F(\lambda) = f(\lambda)\ R$$

or

$$f(\lambda) = F(\lambda) \cdot R^{-1}$$

Substitution of the above transformation identity into equation (31) results in:

$$R\left[F'(\lambda)R^{-1} \cdot F''(\lambda)R^{-1} - F(\lambda)R^{-1} \cdot F'''(\lambda)R^{-1}\right] + F''''(\lambda)R^{-1} = 0$$

or, expressed more simply,

$$F'(\lambda)F''(\lambda) - F(\lambda)F'''(\lambda) + F''''(\lambda) = 0 \qquad (33)$$

The boundary conditions transform to:

(A) $$f(0) = F(0) \cdot R^{-1} = 0$$

or,

$$F(0) = 0 \qquad (34\text{-a})$$

(B) $$f''(0) = F''(0) \cdot R^{-1} = 0$$

or,

$$F''(0) = 0 \tag{34-b}$$

(C) $$f'(\pm 1) = F'(\pm 1) \cdot R^{-1} = 0$$

or,

$$F'(\pm 1) = 0 \tag{34-c}$$

(D) $$f(1) = F(1) \cdot R^{-1} = 1$$

or,

$$F(1) = R \tag{34-d}$$

Although equation (33) is still a difficult, non-linear differential equation, it is now possible to solve it.

It is apparent that the parameter, wall Reynolds number, R, has been transformed from the differential equation into the boundary conditions, an arrangement which is enormously simpler than the previous one.

Series solution.--Assuming a solution in the form of a power series expanded about $\lambda = 0$ yields:

$$F(\ ) = C_o + C_1\lambda + C_2\lambda^2 + C_3\lambda^3 + C_4\lambda^4 + \text{---------} + C_n\lambda^n + \text{---} \tag{35}$$

The boundary conditions can now be applied to reduce the number of coefficients before substituting into the differential equation.

From boundary condition (34-a),

$$F(0) = 0 = C_o$$

Therefore,

$$C_o = 0$$

From boundary condition (34-c),

$$F'(\pm 1) = 0 = C_1 + 2C_2(\pm 1) + 3C_3(\pm 1)^2 + 4C_4(\pm 1)^3 + \text{-----} + nC_n(\pm 1)^{n-1} + \text{---}$$

This is the symmetry condition about $\lambda = 0$, therefore the even coefficients vanish in order to satisfy the $\lambda = +1$ and $\lambda = -1$ condition simultaneously. Or, that is,

$$0 = C_o = C_2 = C_4 = C_6 = C_8 = \text{----------} = C_n = \text{---} \quad n\text{-even}$$

The series then reduces to:

$$F(\lambda) = C_1\lambda + C_3\lambda^3 + C_5\lambda^5 + C_7\lambda^7 + \text{-------} + C_n\lambda^n + \text{---} \quad n\text{-odd} \tag{36}$$

The other boundary conditions where $C_n = C_n$ (R) do not yield any pertinent information at the present. However, it can be seen by inspection that the boundary condition $F''(0) = 0$ is satisfied. The fourth boundary condition $F(1) = R$ is satisfied by choosing proper values of the constants in the series.

In order to determine the coefficients, the assumed series (36) must be substituted into the differential equation (33). This operation is carried out in Appendix A.

The result of this operation is:

$$C_n = \sum_{k=1}^{k=(n-1)} [n - (2K + 3)]\, C_K \cdot$$

$$\cdot \left\{ \frac{[n - (k + 1)]\,[n - (k + 2)]}{n\,(n - 1)\,(n - 2)\,(n - 3)} \right\} C_{n-(k+1)} \qquad \begin{matrix} k\text{-odd} \\ n\text{-odd} \end{matrix} \qquad (37)$$

At this point, it is convenient to transform back to the f-plane, using the previously defined transformation. The result of this transformation is:

$$f(\lambda) = K_1\lambda + K_3\lambda^3 + K_5\lambda^5 + K_7\lambda^7 + \text{-------} + K_n\lambda^n \text{---} \quad n\text{-odd} \quad (38)$$

where, $K_n = C_n \cdot R^{-1}$,

or

$$K_n = R \sum_{i=1}^{i=(n-4)} [n - 2i - 3]\; K_i \left\{ \frac{[n - i - 1][n - i - 2]}{n(n - 1)(n - 2)(n - 3)} \right\} K_{n-i-1} \qquad \begin{matrix} n\text{-odd} \\ i\text{-odd} \end{matrix} \qquad (39)$$

and, $K_n = K_n(R)$.

Here the convergence of the series defined by (39) will be examined. For illustrative purposes, the first few terms of (39) can be listed:

$$K_1 = K_1$$

$$K_3 = K_3$$

$$K_5 = 0$$

$$K_9 = \frac{K_1 K_7 R}{18}$$

$$K_{11} = \frac{K_3K_7R}{165} + \frac{3K_1K_9R}{55}$$

$$K_{13} = \frac{2K_3K_9R}{143} + \frac{2K_1K_9R}{39}$$

$$K_{15} = \frac{10K_3K_{11}R}{546} - \frac{K_7^2R}{390} + \frac{K_1K_3R}{21}$$

In general, the constant K_n consists of $(n-3)/4$ terms, where any left over fractions from the expression $(n-4)/4$ are dropped. Particular attention should be paid to the fact that each coefficient depends upon <u>all</u> previous coefficients and can be expressed in terms of only K_1 and K_3.

Afer examining in detail the formulation of the coefficients, it is noticed that even though each coefficient K_n contains $(n-3)/4$ terms, a <u>single</u> term comprises most of its magnitude. In fact, <u>more than half</u> of the magnitude of K_n is contained in this single term. After this observation, K_n can be written as:

$$\left|K_n\right| = \left|\frac{(n-5)(n-2)(n-3)\,K_1\,K_{n-2}\,R}{n\,(n-1)(n-2)(n-3)} + (\text{Other Terms})\right| \qquad (40)$$

$$\leq \left|2\,\frac{(n-5)\,K_1\,K_{n-2}\cdot R}{(n)(n-1)}\right|$$

Now the ratio test can be applied to the series. It should be pointed out that the ratio test is usually applied by writing down the K_j and K_{j-1} terms and taking the limit of their ratio, but since in this case it is seen that $j = n$ and $j-1 = n-2$, the ratio can be taken directly from equation (40).

Thus:

$$\left| \frac{K_{n-}}{K_{n-2}} \right| \leq \left| (2) \cdot \frac{(n - 5)}{n(n - 1)} \cdot K_1 \cdot R \right|$$

where λ has been chosen at its maximum value of one.

The ratio test requires that the limit of the absolute value of this ratio be less than one for absolute convergence. Then:

$$\lim_{n \to \infty} \left| \frac{K_{n-}}{K_{n-2}} \right| \leq \lim_{n \to \infty} \left| \frac{2(n - 5)\, K_1\, R}{n(n - 1)} \right| = 0 \qquad (41)$$

since K_1 is always less than 1.6, from the study of limiting cases made previously.

Since the limit of (41) is zero, the series given by equation (38) is convergent for all values of the Reynolds number and completely describes the function $f(\lambda)$.

CHAPTER IV

DISCUSSION

A comparison of the IBM 650 computer solution and the series solution is given in Appendix B. It appears that, for Reynolds numbers greater than about 10, the number of terms necessary to cause the series to converge to a value having an error less than 0.1 % is numerically equal to the Reynolds number under investigation. The constants K_1 and K_2 used in the series are evaluated from $f'(o) = K_1$ and $f'''(o) = K_3 \times 6$. It might be noted that, in effect, this is merely choosing values of K_1 and K_3 that will satisfy the boundary condition $f(1) = 1$ and $f'(1) = 0$.

The derivation of only the differential equation determining the flow for porous channels has been presented but the analysis for semi-porous channels and porous pipes is very similar. The series solution of the differential equation for the semi-porous channel and porous pipes is presented in Appendix C and Appendix D respectively.

A short discussion of further work, in particular computer solutions and general analysis, is included in Appendix E.

APPENDIX A

SERIES FORMULATION (Channel)

The first step in the determination of the n-th term of the series is carried out by substituting the assumed series into the differential equation and equating coefficients.

In order to facilitate the substitution, the first four derivatives, along with $F'(\lambda)F''(\lambda)$ and $-F(\lambda)F'''(\lambda)$ will be written out.

$$F(\lambda) = C_1 \quad + C_3\lambda^3 + C_5\lambda^5 + C_7\lambda^7 + C_9\lambda^9 + C_{11}\lambda^{11} + \text{---} + C_n\lambda^n + \text{---}$$

n-odd

$$F'(\lambda) = C_1 + 3C_3\lambda^2 + 5C_5\lambda^4 + 7C_7\lambda^6 + 9C_9\lambda^8 + 11C_{11}\lambda^{10} + 13C_{13}\lambda^{12}\text{---}$$
$$nC_n\lambda^{n-1} + \text{---}$$

$$F''(\lambda) = 3\cdot 2C_3\lambda + 5\cdot 4C_5\lambda^3 + 7\cdot 6C_7\lambda^5 + 9\cdot 8C_9\lambda^7 + \text{---}n(n-1)C_n\lambda^{n-2} + \text{--}$$

$$F'''(\lambda) = 3\cdot 2\cdot 1C_3 + 5\cdot 4\cdot 3C_5\lambda^2 + 7\cdot 6\cdot 5C_7\lambda^4 + 9\cdot 8\cdot 7C_9\lambda^6 + \text{---} +$$
$$+ \text{---} + (n)(n-1)(n-2)C_n\lambda^{n-3} + \text{---}$$

$$F''''(\lambda) = 5\cdot 4\cdot 3\cdot 2C_5\lambda + 7\cdot 6\cdot 5\cdot 4C_7\lambda^3 + 9\cdot 8\cdot 7\cdot 6C_9\lambda^5 + 11\cdot 10\cdot 9\cdot 8C_{11}\lambda^7 +$$
$$+ \text{---} + \text{---} + (n)(n-1)(n-2)(n-3)C_n\lambda^{n-4} + \text{---}$$

Carrying out the multiplications indicated in the differential equation yields:

$$F'(\lambda)F''(\lambda) = (C_1\cdot 3\cdot 2\cdot C_3)\lambda + (3\cdot 2\cdot C_2\cdot 3\cdot C_3 + C_1\cdot 5\cdot 4\cdot C_5)\lambda^3 +$$

$$+ (3\cdot 2\cdot C_3\cdot 5C_5 + 5\cdot 4\cdot C_5\cdot 3\cdot C_3 + 7\cdot 6\cdot C_7\cdot C_1)\lambda^5 + (3\cdot 2\cdot C_3 7C_7 +$$

$$+ 5C_5\cdot 5\cdot 4\cdot C_5 + 7\cdot 6C_7\cdot 3C_3 + 9\cdot 8C_9\cdot C_1)\lambda^7 + (3\cdot 2\cdot C_3\cdot 9\cdot C_9 +$$

$$+ 5\cdot 4\cdot C_5 7C_7 + 7\cdot 6\cdot C_7\cdot 5\cdot C_5 + 9\cdot 8\cdot C_9\cdot 3\cdot C_3 + 11\cdot 10\cdot C_{11}C_1)\lambda^9 +$$

$$+ (3\cdot 2C_3\cdot 11C_{11} + 5\cdot 4C_5\cdot 9\cdot C_9 + 7\cdot 6\cdot C_7\cdot 7C_7 + 9\cdot 8\cdot C_9 5\cdot 6\cdot C_5 +$$

$$+ 11\cdot 10C_{11}\cdot 3C_3 + 13\cdot 12C_{13}\cdot C_1)\lambda^{11} + \text{---}$$

$$-F(\lambda)F'''(\lambda) = -(3\cdot 2\cdot 1\cdot C_3\cdot C_1)\lambda - (3\cdot 2\cdot 1\cdot C_3\cdot C_3 + 5\cdot 4\cdot 3\cdot C_5\cdot C_1)\lambda^3 -$$

$$- (3\cdot 2\cdot 1\cdot C_3\cdot C_5 + 5\cdot 4\cdot 3\cdot C_5\cdot C_3 + 7\cdot 6\cdot 5\cdot C_7\cdot C_1)\lambda^5 -$$

$$- (3\cdot 2\cdot 1\cdot C_3\cdot C_7 + 5\cdot 4\cdot 3C_3\cdot C_5 + 7\cdot 6\cdot 5\cdot C_7\cdot C_3 + 9\cdot 8\cdot 7\cdot C_9\cdot C_1)\lambda^7 -$$

$$- (3\cdot 2\cdot 1\cdot C_3\cdot C_{11} + 5\cdot 4\cdot 3\cdot C_5\cdot C_9 + 7\cdot 6\cdot 5\cdot C_7\cdot C_7 + 9\cdot 8\cdot 7\cdot C_9\cdot C_5 +$$

$$+ 11\cdot 10\cdot 9\cdot C_{11}\cdot C_3 + 13\cdot 12\cdot 11\cdot C_3\cdot C_1)\lambda^9 + \text{---}$$

Substituting into the differential equation,

$$F'(\lambda)F''(\lambda) - F(\lambda)F'''(\lambda) + F''''(\lambda) = 0$$

and equating the various powers of λ to zero after grouping yields:

For λ'

$$3\cdot 2\cdot C_3 C_1 - 3\cdot 2\cdot C_3 C_1 + 5\cdot 4\cdot 3\cdot C_5 = 0$$

For λ^3

$$3\cdot 2\cdot C_3 3C_3 + 5\cdot 4\cdot C_5\cdot C_1 - 3\cdot 2\cdot 1\cdot C_3 C_3 - 5\cdot 4\cdot 3\cdot C_5 C_1 + 7\cdot 6\cdot 5\cdot 4\cdot C_7 = 0$$

For λ^5

$$3\cdot2\cdot C_3\cdot5C_5 + 5\cdot4\cdot C_5\cdot3C_3 + 7\cdot6\cdot C_1\cdot C_1 - 3\cdot2\cdot1C_3\cdot C_5 - 5\cdot4\cdot3\cdot C_5\cdot C_3 -$$
$$- 7\cdot6\cdot5\cdot C_7\cdot C_1 + 9\cdot8\cdot7\cdot6\cdot C_9 = 0$$

For λ^7

$$3\cdot2\cdot C_3\cdot7C_7 + 5\cdot4\cdot C_5\cdot5C_5 + 7\cdot6\cdot C_7\cdot3C3 + 9\cdot8\cdot C_9\cdot C_1 - 3\cdot2\cdot1C_3\cdot C_7 -$$
$$- 5\cdot4\cdot3C_5\cdot C_5 - 7\cdot6\cdot5\cdot C_7\cdot C_3 - 9\cdot8\cdot7\cdot C_9\cdot C_1 +$$
$$+ 11\cdot10\cdot9\cdot8\cdot C_{11} = 0$$

For λ^9

$$3\cdot2\cdot C_3\cdot9C_9 + 5\cdot4\cdot C_5\cdot7C_7 + 7\cdot6\cdot C_7\cdot5\cdot C_3 + 9\cdot8\cdot C_9\cdot3C_3 + 11\cdot10\cdot C_{11}\cdot C_1 -$$
$$- 3\cdot2\cdot1\cdot C_3C_{11} - 5\cdot4\cdot3\cdot C_5C_7 - 7\cdot6\cdot5\cdot C_7C_5 - 9\cdot8\cdot7\cdot C_9C_3 -$$
$$- 11\cdot10\cdot9C_{11}\cdot C_1 + 13\cdot12\cdot11\cdot10\cdot C_{13} = 0$$

For λ^{11}

$$3\cdot2\cdot C_3\cdot11C_9 + 5\cdot4\cdot C_5\cdot9C_9 + 7\cdot6\cdot C_7\cdot7C_7 + 9\cdot8C_9\cdot5C_5 + 11\cdot10\cdot C_{11}\cdot3C_3 +$$
$$+ 13\cdot12\cdot C_{13}C_1 - 3\cdot2\cdot C_3\cdot C_{13} - 5\cdot4\cdot3\cdot C_5\cdot C_9 - 7\cdot6\cdot5C_7\cdot C_7 -$$
$$- 9\cdot8\cdot7\cdot C_9\cdot C_5 - 11\cdot10\cdot9\cdot C_{11}\cdot C_3 - 13\cdot12\cdot11\cdot C_{13}\cdot C_1 +$$
$$+ 15\cdot14\cdot13\cdot12\cdot C_{15} = 0$$

The next step is to solve the above equations for the constants with the highest subscript. The result of this operation is given on the following page.

C_1

C_3

$$C_5 = C_1\left(\frac{-3\cdot2\cdot C_3}{5\cdot4\cdot3\cdot2}\right) + C_3\left(\frac{3\cdot2\cdot1\cdot C_1}{5\cdot4\cdot3\cdot2}\right).$$

$$C_7 = 3C_1\left(\frac{-3\cdot2\cdot C_3}{7\cdot6\cdot5\cdot4}\right) + C_1\left(\frac{-5\cdot4\cdot C_5}{7\cdot6\cdot5\cdot4}\right) + C_3\left(\frac{3\cdot2\cdot1\cdot C_3}{7\cdot6\cdot5\cdot4}\right) + C_1\left(\frac{5\cdot4\cdot3\cdot C_5}{7\cdot6\cdot5\cdot4}\right).$$

$$C_9 = 5C_5\left(\frac{-3\cdot2\cdot C_3}{9\cdot8\cdot7\cdot6}\right) + 3C_3\left(\frac{-5\cdot4\cdot C_5}{9\cdot8\cdot7\cdot6}\right) + C_1\left(\frac{-7\cdot6\cdot C_7}{9\cdot8\cdot7\cdot6}\right) + C_5\left(\frac{3\cdot2\cdot1\cdot C_3}{9\cdot8\cdot7\cdot6}\right) + \\ + C_3\left(\frac{5\cdot4\cdot3\cdot C_5}{9\cdot8\cdot7\cdot6}\right) + C_1\left(\frac{7\cdot6\cdot5\cdot C_7}{9\cdot8\cdot7\cdot6}\right).$$

$$C_{11} = 7C_7\left(\frac{-3\cdot2\cdot C_3}{11\cdot10\cdot9\cdot8}\right) + 5C_5\left(\frac{-5\cdot4\cdot C_5}{11\cdot10\cdot9\cdot8}\right) + 3C_3\left(\frac{-7\cdot6\cdot C_7}{11\cdot10\cdot9\cdot8}\right) + \\ + C_1\left(\frac{-9\cdot8\cdot C_9}{11\cdot10\cdot9\cdot8}\right) + C_7\left(\frac{3\cdot2\cdot1\cdot C_3}{11\cdot10\cdot9\cdot8}\right) + C_5\left(\frac{5\cdot4\cdot3\cdot C_5}{11\cdot10\cdot9\cdot8}\right) + \\ + C_3\left(\frac{7\cdot6\cdot5\cdot C_7}{11\cdot10\cdot9\cdot8}\right) + C_1\left(\frac{9\cdot8\cdot7\cdot C_9}{11\cdot10\cdot9\cdot8}\right).$$

$$C_{13} = 9C_9\left(\frac{-3\cdot2\cdot C_3}{13\cdot12\cdot11\cdot10}\right) + 7C_7\left(\frac{-5\cdot4\cdot C_5}{13\cdot12\cdot11\cdot10}\right) + 5C_5\left(\frac{-7\cdot6\cdot C_7}{13\cdot12\cdot11\cdot10}\right) + \\ + 3C_3\left(\frac{-9\cdot8\cdot C_9}{13\cdot12\cdot11\cdot10}\right) + C_1\left(\frac{-11\cdot10\cdot C_{11}}{13\cdot12\cdot11\cdot10}\right) + C_9\left(\frac{3\cdot2\cdot1C_3}{13\cdot12\cdot11\cdot10}\right) + \\ + C_7\left(\frac{5\cdot4\cdot3\cdot C_5}{13\cdot12\cdot11\cdot10}\right) + C_5\left(\frac{5\cdot4\cdot3\cdot C_7}{13\cdot12\cdot11\cdot10}\right) + C_3\left(\frac{9\cdot8\cdot7\cdot C_9}{13\cdot12\cdot11\cdot10}\right) + \\ + C_1\left(\frac{11\cdot10\cdot9\cdot C_{11}}{13\cdot12\cdot11\cdot10}\right).$$

$$C_{15} = 11C_{11}\left(\frac{-3\cdot 2\cdot 1\cdot C_3}{15\cdot 14\cdot 13\cdot 12}\right) + 9C_9\left(\frac{-5\cdot 4\cdot C_5}{15\cdot 14\cdot 13\cdot 12}\right) + 7C_7\left(\frac{-7\cdot 6\cdot C_7}{15\cdot 14\cdot 13\cdot 12}\right) +$$

$$+ 5C_5\left(\frac{-9\cdot 8\cdot C_9}{15\cdot 14\cdot 13\cdot 12}\right) + 3C_3\left(\frac{-11\cdot 10C_{11}}{15\cdot 14\cdot 13\cdot 12}\right) + C_1\left(\frac{-13\cdot 12\cdot C_{13}}{15\cdot 14\cdot 13\cdot 12}\right) +$$

$$+ C_{11}\left(\frac{3\cdot 2\cdot 1\cdot C_3}{15\cdot 14\cdot 13\cdot 12}\right) + C_9\left(\frac{5\cdot 4\cdot 3\cdot C_5}{15\cdot 14\cdot 13\cdot 12}\right) + C_7\left(\frac{7\cdot 6\cdot 5\cdot C_7}{15\cdot 14\cdot 13\cdot 12}\right) +$$

$$+ C_5\left(\frac{9\cdot 8\cdot 7\cdot C_9}{15\cdot 14\cdot 13\cdot 12}\right) + C_3\left(\frac{11\cdot 10\cdot 9\cdot C_{11}}{15\cdot 14\cdot 13\cdot 12}\right) + C_1\left(\frac{13\cdot 12\cdot 11\cdot C_{13}}{15\cdot 14\cdot 13\cdot 12}\right).$$

By collecting terms and writing them in an orderly manner it is possible to deduce the n-th term (i.e., n-th constant term).

C_1

C_3

$$C_5 = -C_1\left(\frac{3\cdot 2\cdot C_3}{5\cdot 4\cdot 3\cdot 2}\right) + C_3\left(\frac{3\cdot 2\cdot 1\cdot C_1}{5\cdot 4\cdot 3\cdot 2}\right) = 0$$

$$C_7 = -2C_3\left(\frac{3\cdot 2\cdot C_3}{7\cdot 6\cdot 5\cdot 4}\right) + 2C_1\left(\frac{-5\cdot 4\cdot C_5}{7\cdot 6\cdot 5\cdot 4}\right)$$

$$C_9 = -4C_5\left(\frac{3\cdot 2\cdot C_3}{9\cdot 8\cdot 7\cdot 6}\right) + 0\cdot C_3\left(\frac{5\cdot 4\cdot C_5}{9\cdot 8\cdot 7\cdot 6}\right) + 4C_1\left(\frac{7\cdot 6\cdot C_7}{9\cdot 8\cdot 7\cdot 6}\right)$$

$$C_{11} = 6C_7\left(\frac{3\cdot 2\cdot C_3}{11\cdot 10\cdot 9\cdot 8}\right) - 2C_5\left(\frac{5\cdot 4\cdot C_5}{11\cdot 10\cdot 9\cdot 8}\right) + 2C_3\left(\frac{7\cdot 6\cdot C_7}{11\cdot 10\cdot 9\cdot 8}\right) +$$

$$+ 6C_1\left(\frac{9\cdot 8\cdot C_9}{11\cdot 10\cdot 9\cdot 8}\right)$$

$$C_{13} = -8C_9\left(\frac{3\cdot 2\cdot C_3}{13\cdot 12\cdot 11\cdot 10}\right) - 4C_7\left(\frac{5\cdot 4\cdot C_5}{13\cdot 12\cdot 11\cdot 10}\right) + 0C_5\left(\frac{7\cdot 6\cdot C_7}{13\cdot 12\cdot 11\cdot 10}\right) + 4C_3\left(\frac{9\cdot 8\cdot C_9}{13\cdot 12\cdot 11\cdot 10}\right) + 8C_1\left(\frac{11\cdot 10\cdot C_{11}}{13\cdot 12\cdot 11\cdot 10}\right)$$

$$C_{15} = -10C_{11}\left(\frac{3\cdot 2\cdot C_3}{15\cdot 14\cdot 13\cdot 12}\right) - 6C_9\left(\frac{5\cdot 4\cdot C_5}{15\cdot 14\cdot 13\cdot 12}\right) - 2C_7\left(\frac{7\cdot 6\cdot C_7}{15\cdot 14\cdot 13\cdot 12}\right) + 2C_5\left(\frac{9\cdot 8\cdot C_9}{15\cdot 14\cdot 13\cdot 12}\right) + 6C_3\left(\frac{11\cdot 10\cdot C_{11}}{15\cdot 14\cdot 13\cdot 12}\right) + 10C_1\left(\frac{13\cdot 12\cdot C_{13}}{15\cdot 14\cdot 13\cdot 12}\right)$$

or

$$C_n = \sum_{k=1}^{k=n-4} [n - (2k + 3)]C_k \left\{\frac{[n-(k+1)][n-(k+1)]}{n(n-1)\,(n-2)(n-3)}\, C_{n-(k+1)}\right\} \qquad \begin{array}{l} k = \text{odd} \\ n = \text{odd} \end{array}$$

APPENDIX B

COMPARISON OF SERIES AND COMPUTER SOLUTIONS

$R = 10$

		$K_n \lambda^n$				
n	K_n	$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 0.9$	$\lambda = 1.0$
1	1.35530	0.27106	0.67765	0.94871	1.21977	1.35530
3	-0.31328	-0.00251	-0.03916	-0.10746	-0.22838	-0.31328
5	0	0	0	0	0	0
7	-0.01402		-0.00011	-0.00115	-0.00671	-0.01407
9	-0.015056		-0.00002	-0.00043	-0.00409	-0.01056
11	-0.00754			-0.00015	-0.00237	-0.00754
13	-0.00480			-0.00005	-0.00122	-0.00480
15	-0.00267			-0.00001	-0.00055	-0.00267
17	-0.00129				-0.00022	-0.00129
19	-0.00042				-0.00006	-0.00042
21	-0.00022				-0.00002	-0.00022
23	-0.00007					-0.00007
25	-0.00002					-0.00002
27	-0.00001					-0.00001

	$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 0.9$	$\lambda = 1.0$	
Summation (series):	.26855	0.63836	0.83946	0.97615	1.00040	$f(\lambda)$
Computer Results:	.26855	0.63836	0.83946	0.97616	1.00041	$f(\lambda)$

APPENDIX C

THE SEMI-POROUS CHANNEL

The differential equation comparable to equation (31) for the case of a channel with only one wall porous has been derived by Donoughe (5). The transformation previously defined also transforms the parameter R from the equation and into the boundary conditions for this case.

The equation describing the conditions existing in the semi-porous channel is given by:

$$R_s(g'^2 - gg'') + g''' = A$$

where $g \equiv g(\lambda)$. Upon differentiation, the equation becomes:

$$R_s(g'g'' - g''') + g'''' = 0$$

Where the wall Reynolds number, R_s, is in this case given by:

$$R_s \equiv \frac{v_w h}{\nu}$$

When the transformation is effected, the following equation results:

$$G'G'' - G\,G''' + G'''' = 0$$

with the following boundary conditions applying:

$$\lambda = 0\text{: } G = 0;\ G' = 0$$

$$\lambda = 1\text{: } G = R;\ G' = 0$$

Assuming a solution in the form of a power series expanded about $\lambda = 0$ yields:

$$G = C_o + C_1\lambda + C_2\lambda^2 + C_3\lambda^3 + ---- + C_n\lambda^n + ----$$

The boundary conditions can now be employed in order to reduce coefficients as follows.

From $G = 0$ at $\lambda = 0$:

$$C_o = 0$$

From $G' = 0$ at $\lambda = 0$:

$$C_1 = 0$$

The remaining two boundary conditions do not yield any applicable information at this point.

The series can now be written as:

$$G = C_2\lambda^2 + C_3\lambda^3 + C_4\lambda^4 + C_5\lambda^5 + --- + C_n\lambda^n + ---$$

The next step will be to write out the first four derivatives and carry through a calculation similar to calculation for the fully porous channel.

$$G' = 2C_2\lambda + 3C_3\lambda^2 + 4C_4\lambda^3 + 5C_5\lambda^4 + --- + nC_n\lambda^{n-1} + ---$$

$$G'' = 2 \cdot 1C_2 + 3 \cdot 2C_3\lambda + 4 \cdot 3C_4\lambda^2 + 5 \cdot 4C_5\lambda^3 + --- + n(n-1)\lambda^{n-2} + ---$$

$$G''' = 3\cdot2\cdot1C_3 + 4\cdot3\cdot2C_4\lambda + 5\cdot4\cdot3C_5\lambda^2 + --- + n(n-1)(n-2)\lambda^{n-3} + $$
$$+ ---$$

$$G'''' = 4\cdot3\cdot2\cdot1C_4 + 5\cdot4\cdot3\cdot2C_5\lambda + 6\cdot5\cdot4\cdot3C_6\lambda^2 + --- +$$
$$+ n(n-1)(n-2)(n-3)\lambda^{n-4} + ---$$

The multiplications indicated in the transformed differential equation must be carried out before the substitution can be effected. This operation is:

$$G'G'' = (1 - 2C_2\cdot C_2)\lambda + (2C_2\cdot2\cdot3C_3 + 1\cdot2\cdot C_2\cdot3\cdot C_2)\lambda^2 + (1\cdot2\cdot C_2\cdot4C_4 +$$
$$+ 2\cdot3\cdot C_3\cdot3C_3 + 2C_2\cdot3\cdot4\cdot C_4)\lambda^3 + (1\cdot2\cdot C_2\cdot5C_5 + 2\cdot3\cdot C_3\cdot4C_4 +$$
$$+ 3C_3\cdot3\cdot4\cdot C_4 + 4\cdot5\cdot C_5\cdot2C_2)\lambda^4 + (1\cdot2\cdot C_2\cdot6C_6 + 2\cdot3\cdot C_3\cdot5C_5 +$$
$$+ 3\cdot4\cdot C_4\cdot4C_4 + 4\cdot5\cdot C_5\cdot3C_3 + 5\cdot6C_6\cdot2C_2)\lambda^5 + ---$$

$$-GG'' = -(0)\lambda - (2\cdot3\cdot C_3\cdot C_2)\lambda^2 - (2\cdot3\cdot C_3\cdot C_3 + 2\cdot3\cdot4\cdot C_4\cdot C_2)\lambda^3 +$$
$$- (2\cdot3\cdot C_3\cdot C_4 + 2\cdot3\cdot4\cdot C_4\cdot C_3 + 3\cdot4\cdot5\cdot C_5\cdot C_2)\lambda^4 - (2\cdot3\cdot C_3\cdot C_5 +$$
$$+ 2\cdot3\cdot4\cdot C_4\cdot C_4 + 3\cdot4\cdot5\cdot C_5\cdot C_3 + 4\cdot5\cdot6\cdot C_6\cdot C_2)\lambda^5$$

Expressing these results as indicated by the differential equation and equating the coefficients of the corresponding powers of λ to zero yields:

For λ^0:

$$2 \cdot 3 \cdot 4 \cdot C_4 = 0$$

For λ^1:

$$1 \cdot 2 \cdot C_2 \cdot 2C_2 + 2 \cdot 3 \cdot 4 \cdot 5C_5 = 0$$

For λ^2:

$$2 \cdot C_2 \cdot 2 \cdot 3 \cdot C_3 + 1 \cdot 2 \cdot C_2 \cdot 3 \cdot C_3 - 2 \cdot 3 \cdot C_3 \cdot C_2 + 3 \cdot 4 \cdot 5 \cdot 6 \cdot C_6 = 0$$

For λ^3:

$$1 \cdot 2 \cdot C_2 \cdot 4C_4 + 2 \cdot 3 \cdot C_3 \cdot 3C_3 + 2C_2 \cdot 3 \cdot 4 \cdot C_4 - 2 \cdot 3 \cdot C_5 \cdot C_3 \cdot 2 \cdot 3 \cdot 4 \cdot C_4 \cdot C_2 + \\ + 4 \cdot 5 \cdot 6 \cdot 7 \cdot C_7 = 0$$

For λ^4:

$$1 \cdot 2 \cdot C_2 \cdot 5C_5 + 2 \cdot 3 \cdot C_3 \cdot 4C_4 + 3 \cdot C_3 \cdot 3 \cdot 4 \cdot C_4 + 4 \cdot 5 \cdot C_5 \cdot 2C_2 - 2 \cdot 3 \cdot C_3 \cdot C_4 + \\ - 2 \cdot 3 \cdot 4 \cdot C_4 \cdot C_3 - 3 \cdot 4 \cdot 5 \cdot C_5 \cdot C_2 + 5 \cdot 6 \cdot 7 \cdot 8 \cdot C_8 = 0$$

For λ^5:

$$1 \cdot 2 \cdot C_2 \cdot 6C_6 + 2 \cdot 3 \cdot C_3 \cdot 5C_5 + 3 \cdot 4 \cdot C_4 \cdot 4C_4 + 4 \cdot 5 \cdot C_5 \cdot 3C_3 + 5 \cdot 6 \cdot C_6 \cdot 2C_2 + \\ - 2 \cdot 3 \cdot C_3 \cdot C_5 - 2 \cdot 3 \cdot 4 \cdot C_4 \cdot C_4 - 3 \cdot 4 \cdot 5 \cdot C_5 \cdot C_3 \cdot 4 \cdot 5 \cdot 6 \cdot C_6 \cdot C_2 + \\ + 6 \cdot 7 \cdot 8 \cdot 9C_9 = 0$$

Solving the preceding expressions for the coefficient with the highest subscript yields:

C_2

C_3

$C_4 = 0$

$$C_5 = \frac{1}{2\cdot3\cdot4\cdot5}\left\{-\,(1\cdot2\cdot C_2)2C_2\right\}$$

$$C_6 = \frac{1}{3\cdot4\cdot5\cdot6}\left\{-\,2C_2(3\cdot4\cdot C_4) - 3C_3(2\cdot3\cdot C_3) - 4C_4(1\cdot2\cdot C_2) + C_2(2\cdot3\cdot4C_4) + \right.$$
$$\left. + C_3(1\cdot2\cdot3C_3)\right\}$$

$$C_7 = \frac{1}{4\cdot5\cdot6\cdot7}\left\{-\,2C_2(3\cdot4\cdot C_4) - 3C_3(2\cdot3C_3) - 4C_4(1\cdot2C_2) + C_2(2\cdot3\cdot4C_4) + \right.$$
$$\left. + C_3(1\cdot2\cdot3C_3)\right\}$$

$$C_8 = \frac{1}{5\cdot6\cdot7\cdot8}\left\{-\,2\cdot C_2(4\cdot5\cdot C_5) - 3\cdot C_3(3\cdot4\cdot C_4) - 4C_4(2\cdot3\cdot C_3) - 5\cdot C_5(1\cdot2\cdot C_2) + \right.$$
$$\left. + C_2(3\cdot4\cdot5\cdot C_5) + C_3(2\cdot3\cdot4\cdot C_4) + C_4(2\cdot3\cdot C_3)\right\}$$

$$C_9 = \frac{1}{6\cdot7\cdot8\cdot9}\left\{-\,2C_2(5\cdot6C_6) - 3C_3(4\cdot5C_5) - 4C_4(3\cdot4C_4) - 5C_5(2\cdot3C_3) - \right.$$
$$- 6C_6(1\cdot2C_2) + C_2(4\cdot5\cdot6C_6) + C_3(3\cdot4\cdot5C_5) + C_4(2\cdot3\cdot4C_4) +$$
$$\left. + C_5(1\cdot2\cdot3C_3)\right\}$$

or the n-th constant can be written as:

$$C_n = \frac{(n - 4)!}{n!} \left\{ \sum_{K = 2}^{K = n - 3} - K\, C_K [n - (K + 2)][n - (K + 1)] C_{n - (K + 1)} + \right.$$

$$\left. + \sum_{K = 1}^{K = n - 4} C_K [n - (K + 3)][n - (K + 1)][n - (K + 1)] C_{n - (K + 1)} \right\}$$

APPENDIX D

THE POROUS WALL PIPE

The Navier-Stokes Equations reduce, in the case of the circular pipe, to the following differential equation:

$$\eta f''' + f'' - R_p(f'^2 - ff'') = K(R_p)$$

where

$$\eta = \left(\frac{r}{r_o}\right)^2 = \text{dimensionless radius,}$$

and

$$R_p = \frac{v_w r_o}{\nu} = \text{wall Reynolds number.}$$

Upon differentiation, the differential equation becomes:

$$\eta f'''' + 2f''' - R_p(f'f'' - ff''') = 0$$

Application of the transformation results in the following formulation:

$$\eta F'''' + 2F''' - F'F'' + FF''' = 0$$

The boundary conditions for each of these cases are:

$$f(0) = f'(1) = 0 \text{ ------------- } F(0) = F'(1) = 0$$

$$f(1) = \frac{1}{2} \text{ -------------- } F(1) = \frac{1}{2} R_p$$

$$\lim_{\eta \to 0} \sqrt{\eta}\, f''(\eta) = 0 \text{ -------------- } \lim_{\eta \to 0} \sqrt{\eta}\, F''(\eta) = 0$$

Assumption of a solution in the form of a power series expanded about $\eta = 0$ yields:

$$F(\eta) = D_o + D_1\eta + D_2\eta^2 + D_3\eta^3 + D_4\eta^4 + \text{---} + D_n\eta^n + \text{---}$$

The boundary condition $F(0) = 0$, is employed to show that:

$$D_o = 0$$

The remaining boundary conditions do not yield any pertinent information at this point.

The series can now be written as:

$$F(\eta) = D_1\eta + D_2\eta^2 + D_3\eta^3 + D_4\eta^4 + D_5\eta^5 + D_6\eta^6 + \text{---} + D_n\eta^n + \text{---}$$

The differential equation indicates the use of the first four derivatives, which are given by:

$$F'(\eta) = D_1 + 2D_2\eta + 3D_3\eta^2 + 4D_4\eta^3 + 5D_5\eta^4 + \text{---} + nD_n\eta^{n-1} + \text{---}$$

$$F''(\eta) = 1\cdot 2D_2 + 2\cdot 3\cdot D_3\eta + 3\cdot 4D_4\eta^2 + 4\cdot 5D_5\eta^3 + 5\cdot 6D_6\eta^4 + \text{---} + $$

$$+ n(n-1)D_n\eta^{n-2} + \text{---}$$

$$F'''(\eta) = 1\cdot 2\cdot 3D_3 + 2\cdot 3\cdot 4D_4\eta + 3\cdot 4\cdot 5D_5\eta^2 + 4\cdot 5\cdot 6D_6\eta^3 + --- +$$

$$+ n(n - 1)(n - 2)D_n\eta^{n-3} + ---$$

$$F''''(\eta) = 1\cdot 2\cdot 3\cdot 4D_4 + 2\cdot 3\cdot 4\cdot 5D_5\eta + 3\cdot 4\cdot 5\cdot 6D_6\eta^2 + --- +$$

$$+ n(n - 1)(n - 2)(n - 3)D_n\eta^{n-4} + ---$$

The indicated operations with these derivatives are as follows:

$$\eta F'''' = 1\cdot 2\cdot 3\cdot 4D_4\eta + 2\cdot 3\cdot 4\cdot 5D_5\eta^2 + 3\cdot 4\cdot 5\cdot 6D_6\eta^3 + 4\cdot 5\cdot 6\cdot 7D_7\eta^4 + ---$$

$$+2F''' = 2\cdot 1\cdot 2\cdot 3D_3 + 2\cdot 2\cdot 3\cdot 4D_4\eta + 2\cdot 3\cdot 4\cdot 5D_5\eta^2 + 2\cdot 4\cdot 5\cdot 6D_6\eta^3 +$$

$$+ 2\cdot 5\cdot 6\cdot 7D_7\eta^4 + ---$$

$$-F'F'' = - 2D_1D_2 - (2D_2\cdot 2D_2 + D_1\cdot 2\cdot 3D_3)\eta - 2(2D_2\cdot 2\cdot 3D_3 + 2D_2\cdot 3D_3 +$$

$$+ D_1\cdot 3\cdot 4D_4)\eta^2 - (D_1\cdot 4\cdot 5\cdot D_5 + 2D_2\cdot 4D_4 + 2D_2\cdot 3\cdot 4\cdot D_4 +$$

$$+ 2\cdot 3D_3\cdot 3D_3)\eta^3 - (2D_2\cdot 5D_5 + D_1\cdot 5\cdot 6D_6 + 2\cdot 3\cdot D_3\cdot 4D_4 +$$

$$+ 3\cdot 4D_4\cdot 3D_3 + 2D_2\cdot 4\cdot 5D_5)\eta^4 + ---$$

$$FF''' = (D_1\cdot 1\cdot 2\cdot 3D_3)\eta + (1\cdot 2\cdot 3D_3\cdot D_2 + 2\cdot 3\cdot 4D_4\cdot D_1)\eta^2 + (1\cdot 2\cdot 3D_3\cdot D_3 +$$

$$+ 2\cdot 3\cdot 4D_4\cdot D_2 + 3\cdot 4\cdot 5\cdot D_5\cdot D_1)\eta^3 + (1\cdot 2\cdot 3\cdot D_3\cdot D_4 + 2\cdot 3\cdot 4\cdot D_4\cdot D_3 +$$

$$+ 3\cdot 4\cdot 5D_5\cdot D_2 + 4\cdot 5\cdot 6D_6\cdot D_1)\eta^4 + ----$$

Equating the coefficients of the various powers of η to zero as indicated by the differential equation yields:

For η^0

$$1\cdot 2\cdot 3D_3 - 2D_1\cdot D_2 = 0$$

For η^1

$$1\cdot 2\cdot 3\cdot 4D_4 + 2\cdot 2\cdot 3\cdot 4D_4 - 2\cdot D_2\cdot 2D_2 - D_1\cdot 2\cdot 3\cdot D_3 + D_1\cdot 1\cdot 2\cdot 3D_3 = 0$$

For η^2

$$2\cdot 3\cdot 4\cdot 5D_5 + 2\cdot 3\cdot 4\cdot 5D_5 - 2\cdot D_2\cdot 2\cdot 3D_3 - 2\cdot D_3\cdot 3D_3 - D_1\cdot 3\cdot 4\cdot D_4 + $$
$$+ 1\cdot 2\cdot 3\cdot D_3\cdot D_2 + 2\cdot 3\cdot 4D_4\cdot D_1 = 0$$

For η^3

$$3\cdot 4\cdot 5\cdot 6D_6 + 2\cdot 4\cdot 5\cdot 6\cdot D_7 - D_1\cdot 4\cdot 5\cdot D_5 - 2D_2\cdot 4D_4 - 2D_2\cdot 3\cdot 4D_4 -$$
$$- 2\cdot 3D_3\cdot 3D_3 + 1\cdot 2\cdot 3D_3\cdot D_3 + 2\cdot 3\cdot 4D_4\cdot D_2 + 3\cdot 4\cdot 5D_5\cdot D_1 = 0$$

For η^4

$$4\cdot 5\cdot 6\cdot 7D_7 + 2\cdot 5\cdot 6\cdot 7D_7 - 2D_2\cdot 5D_5 - D_1\cdot 5\cdot 6D_6\cdot 2\cdot 3D_3\cdot 4D_4 + 3\cdot 4D_4\cdot 3D_3 +$$
$$+ 2D_2\cdot 4\cdot 5D_5 + 1\cdot 2\cdot 3D_3\cdot D_4 + 2\cdot 3\cdot 4D_4\cdot D_3 + 3\cdot 4\cdot 5D_5\cdot D_2 +$$
$$+ 4\cdot 5\cdot 6\cdot D_6\cdot D_1 = 0$$

After adding like terms and rearranging, the following formulation results:

D_1

D_2

$$D_3 = \frac{1}{2(1\cdot 2\cdot 3)} \left\{ \left[2D_1 \cdot D_2 \right] \right\}$$

$$D_4 = \frac{1}{3(2\cdot 3\cdot 4)} \left\{ \left[2D_2 \cdot 2D_2 \right] \right\}$$

$$D_5 = \frac{1}{4(3\cdot 4\cdot 5)} \left\{ \left[2D_2 \cdot 3D_3 \right] + \left[D_2 \cdot 2\cdot 3D_3 - D_1 \cdot 3\cdot 4D_4 \right] \right\}$$

$$D_6 = \frac{1}{5(4\cdot 5\cdot 6)} \left\{ \left[2D_2 \cdot 4D_4 \right] + \left[2D_3 \cdot 2\cdot 3D_3 - 2D_1 \cdot 4\cdot 5D_5 \right] \right\}$$

$$D_7 = \frac{1}{6(5\cdot 6\cdot 7)} \left\{ \left[2D_2 \cdot 4D_4 \right] + \left[3D_4 \cdot 2\cdot 3\cdot D_3 + D_3 \cdot 3\cdot 4D_4 - D_2 \cdot 4\cdot 5D_5 - 3D_1 \cdot 5\cdot 6D_6 \right] \right\}$$

or the n-th constant can be written as:

$$D_n = \frac{1}{n(n-1)^2(n-2)} \left\{ \left[2D_2 \, (n-2) D_{n-2} \right] + \sum_{K=1}^{K=(n-3)} [n - 2K - 2]\, D_{n-K-2}\, [n + K - 5][n + K - 4]\, D_{n+K-4} \right\}$$

APPENDIX E

REFERENCE TO FURTHER WORK

It has been pointed out previously that this work was performed in conjunction with that of Mr. Frank M. White, Jr., and that only the work on the series solution was performed entirely by the author. For this reason a brief statement of the contents of Mr. White's unpublished thesis will be given.

As a sequel to the discussion given on the channel in this thesis, Mr. White's work contains a presentation of the limiting solutions, the velocity profiles, pressure distribution, friction coefficient and IBM computer solution. Mr. White's thesis also contains a discussion of the semi-porous channel and the porous pipe, along with the above-named information concerning these cases.

APPENDIX F

LIST OF SYMBOLS

C	constant
D	constant
f	dimensionless ratio $\phi(\lambda)/\phi(1)$
F	transformed f, $F = fR$
g	functional notation for semi-porous channel
G	transformed g, $G = gR$
h	half of the channel width for porous channel, total channel width for semi-porous channel.
K	constant depending upon wall Reynolds number.
L	length of porous channel, $L = \frac{\overline{u}(0)\ h}{v_w}$
p, P	pressure
r	radius variable
r_o	radius of the porous wall pipe
R	wall Reynolds number, for fully porous channel, equals $\frac{v_w h}{\nu}$
R_s	wall Reynolds number for semi-porous channel, equals $\frac{v_w h}{\nu}$
R_p	wall Reynolds number of porous pipe, equals $\frac{v_w\ r_o}{\nu}$
u	velocity in the x direction
$u(0)$	average velocity in the x direction at the entrance
v	velocity in the y (or λ) direction
v_w	velocity through the porous walls

w	velocity in the z direction
x,y,z,	Cartesian coordinates
X,Y,Z,	body forces in the corresponding directions
ρ	density
λ	dimensionless coordinate y/h
η	dimensionless coordinate $(r/r_o)^2$
μ	dynamic viscosity
ν	kinematic viscosity, $\nu = \mu/\rho$
ψ	stream function
ϕ	functional notation
$\sum$	summation notation
$'$, $''$, $'''$, $''''$	derivatives (1st, 2nd, 3rd, 4th)

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